

STA 732: Homework 8

March 14, 2026

Instructions

Complete **4 out of 5** problems. You may choose which problem to omit. Clearly indicate which four you chose.

Problem 1

Solve Exercise 4.4 from the lecture notes.

Problem 2

Solve Exercise 4.5 from the lecture notes.

Problem 3

Solve Exercise 4.8 from the lecture notes.

Problem 4

Let X_1, \dots, X_n be i.i.d. observations and consider testing

$$H_0 : \mu = \mu_0 \quad \text{versus} \quad H_1 : \mu \neq \mu_0,$$

where $\mu = \mathbb{E}[X_1]$ and $\mu_0 = 2$, at nominal level $\alpha = 0.05$. The two-sided t -test rejects H_0 when

$$|T_n| := \left| \frac{\sqrt{n}(\bar{X}_n - \mu_0)}{S_n} \right| > t_{n-1, 1-\alpha/2},$$

where $t_{n-1, 1-\alpha/2}$ is the $(1 - \alpha/2)$ -quantile of the t -distribution with $n - 1$ degrees of freedom.

We examine three families of distributions, each with mean $\mu_0 = 2$ under H_0 :

- $N(\mu, \sigma^2)$: the normal distribution with mean $\mu > 0$ and unknown variance $\sigma^2 > 0$;
- $\text{Exp}(\mu)$: the exponential distribution with mean $\mu > 0$;
- $\text{Pareto}(\kappa, 1)$: the Pareto distribution with shape $\kappa > 0$ and known lower bound $x_m = 1$, having density $f(x; \kappa) = \kappa x^{-(\kappa+1)}$ for $x \geq 1$. The mean is $\mu = \kappa/(\kappa - 1)$ for $\kappa > 1$, so $H_0 : \mu = 2$ corresponds to $\kappa_0 = 2$.

- (a) For each family, simulate the actual rejection probability of $|T_n| > t_{n-1, 1-\alpha/2}$ under H_0 for sample sizes $n \in \{50, 100, 500, 1000\}$, using at least 10,000 replications. Report your results in a table, akin to Example 6.1 in the notes. Comment on what you observe: for which families does the t -test seem to achieve level α exactly?
- (b) Cite results from the notes to argue why the t -test is UMPU for the normal family. Then derive the UMPU tests for the exponential and Pareto families. You can either express the critical values as quantile conditions on chi-squared distributions or use a numerical approach to find the critical values. (*Hint*: for the former; you may use without proof that if $T \sim \text{Gamma}(n, \mu_0)$, then for any $c > 0$,

$$\mathbb{E}_{\mu_0}[T \cdot \mathbf{1}\{T > c\}] = n\mu_0 \cdot P_{\mu_0}(T' > c),$$

where $T' \sim \text{Gamma}(n+1, \mu_0)$.)

- (c) For the exponential and Pareto families, compare the power of the UMPU test from (b) at level 0.05 to the power of the (misspecified) t -test at local alternatives $\mu_n = \mu_0 + c/\sqrt{n}$, for a fixed constant $c > 0$ of your choice, for $n \in \{50, 100, 500, 1000\}$. Report your results in a table or figure.
- (d) Interpret your findings. Does the power gap between the UMPU test and the t -test close as n grows for the exponential distribution? What about the Pareto? What does this suggest about the t -test as a general-purpose procedure, if we are not sure about the family of distributions underlying the data?

Problem 5

Let $X_1, \dots, X_n \stackrel{\text{iid}}{\sim} \text{Exp}(\mu)$ with density

$$p_\mu(x) = \frac{1}{\mu} e^{-x/\mu} \mathbf{1}\{x \geq 0\}, \quad \mu > 0.$$

Consider the MLE $\hat{\mu}_n = \bar{X}_n$ and the generic MLE-based confidence set from Example 6.2.

- (a) Following the logic of Example 6.2, show that the corresponding asymptotic $(1-\alpha)$ -confidence interval using the MLE is

$$C_n^{(\mu)}(X) := \left\{ \mu > 0 : n \frac{(\bar{X}_n - \mu)^2}{\mu^2} \leq z_{1-\alpha/2}^2 \right\}.$$

with coverage probability

$$P_\mu(\mu \in C_n^{(\mu)}(X)) = P\left(2n\left(1 - \frac{z_{1-\alpha/2}}{\sqrt{n}}\right) \leq \chi_{2n}^2 \leq 2n\left(1 + \frac{z_{1-\alpha/2}}{\sqrt{n}}\right)\right).$$

- (b) Show that the expected lengths satisfies

$$E_\mu[|C_n^{(\mu)}(X)|] = \mu \frac{2z_{1-\alpha/2}/\sqrt{n}}{1 - z_{1-\alpha/2}^2/n}.$$

(c) Consider confidence intervals of the form

$$C_n^{(a,b)}(X) = \left[\frac{2n\bar{X}_n}{b}, \frac{2n\bar{X}_n}{a} \right]$$

Among all such intervals such that $P(\chi_{2n}^2 \in [a, b]) = 1 - \alpha$, argue that the expected length

$$E_\mu[|C_n^{(a,b)}(X)|] = 2n\mu \left(\frac{1}{a} - \frac{1}{b} \right)$$

is minimized by solving

$$\min_{a,b} \frac{1}{a} - \frac{1}{b} \quad \text{subject to} \quad F_{2n}(b) - F_{2n}(a) = 1 - \alpha,$$

where F_{2n} denotes the χ_{2n}^2 CDF.

(d) For $n \in \{5, 10, 20, 50, 100\}$ and $1 - \alpha = 0.95$, solve the two-equation system from part (c) numerically to obtain the optimal (a_n^*, b_n^*) , and define

$$C_n^{\text{opt}}(X) = \left[\frac{2n\bar{X}_n}{b_n^*}, \frac{2n\bar{X}_n}{a_n^*} \right].$$

Report the expected lengths of C_n^{opt} and $C_n^{(\mu)}$ side by side. How large is the gain from optimization over equal-tails at small n ? Does it vanish as $n \rightarrow \infty$? Interpret your findings.