Part III

Appendix

A Metric Spaces

A.1 Metrics

Definition A.1 (Metric)**.** Let X be a set. A *metric* on X is a function $d : X \times X \to \mathbb{R}$ such that for all $x, y, z \in X$:

(i) $d(x, y) \geqslant 0$ (non-negativity);

(ii) $d(x, y) = 0$ if and only if $x = y$ (identity of indiscernibles);

(iii) $d(x, y) = d(y, x)$ (symmetry);

(iv) $d(x, z) \leqslant d(x, y) + d(y, z)$ (triangle inequality).

Definition A.2 (Metric Space)**.** A *metric space* is a pair (X, d), where X is a set and d is a metric on X.

Example A.3 (Euclidean Space)**.** Let $X = \mathbb{R}^n$. The Euclidean metric is defined by

$$d(x, y) = \|x - y\|_2 = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}.$$

Then $(\mathbb{R}^n, d)$ is a metric space. ♢

Example A.4 (Function Space)**.** Let $X = C[0, 1]$, the set of continuous real-valued functions on the interval $[0, 1]$. The supremum metric (or uniform metric) is defined by

$$d(f, g) = \sup_{t \in [0,1]} |f(t) - g(t)|.$$

Then $(C[0, 1], d)$ is a metric space. ♢

A.2 Topology

Definition A.5 (Open Ball)**.** Let (X, d) be a metric space. The *open ball* of radius $r > 0$ centered at $x \in X$ is the set

$$B_r(x) = \{y \in X : d(x, y) < r\}.$$

Definition A.6 (Open Set in Metric Spaces)**.** A subset $U \subseteq X$ is called *open* if for every $x \in U$, there exists an $\epsilon > 0$ such that $B_\epsilon(x) \subseteq U$.

Definition A.7 (Neighborhood)**.** A subset $N \subseteq X$ is called a *neighborhood* of a point $x \in X$ if there exists an open set U such that $x \in U \subseteq N$. Equivalently, N is a neighborhood of x if there exists an $\epsilon > 0$ such that $B_\epsilon(x) \subseteq N$.

Proposition A.8. *Let (X, d) be a metric space. The collection $\mathcal{T}$ of open sets in X (as defined in Definition A.6) satisfies the following properties:*

(i) $\varnothing \in \mathcal{T}$ and $X \in \mathcal{T}$;

(ii) The union of any collection of open sets is open;

(iii) The intersection of any finite collection of open sets is open.

Definition A.9 (Continuous Function)**.** Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f : X \to Y$ is *continuous* at a point $x \in X$ if for every $\epsilon > 0$, there exists a $\delta > 0$ such that $d_X(x, y) < \delta$ implies $d_Y(f(x), f(y)) < \epsilon$. The function f is *continuous* if it is continuous at every point in X.

Proposition A.10. *Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f : X \to Y$ is continuous if and only if for every open set $V \subseteq Y$, the preimage $f^{-1}(V)$ is open in X.*

Proposition A.10 reveals that continuity can be characterized entirely in terms of open sets, without explicit reference to the underlying metric.

Definition A.11 (Topology Generated by a Metric)**.** The collection of all open sets in a metric space (X, d) forms a topology on X, called the *topology induced by the metric d*.

This motivates the generalization of continuity in metric spaces to spaces where only the notion of "openness" is defined, which leads to the definition of a topological space. It turns out that the properties of Proposition A.8 are precisely the properties needed to have things function the way they do for metrics.

Definition A.12 (Topology)**.** A *topology* on a set X is a collection $\mathcal{T}$ of subsets of X satisfying:

(i) $\varnothing \in \mathcal{T}$ and $X \in \mathcal{T}$;

(ii) The union of any collection of sets in $\mathcal{T}$ is in $\mathcal{T}$;

(iii) The intersection of any finite collection of sets in $\mathcal{T}$ is in $\mathcal{T}$.

The pair $(X, \mathcal{T})$ is called a *topological space*. The elements of $\mathcal{T}$ are called *open sets*.

Remark A.13. Every metric induces a topology, but not every topology arises from a metric. A topological space whose topology is induced by a metric is called *metrizable*.

Definition A.14 (Separable Space)**.** A topological space X is called *separable* if it contains a countable dense subset. That is, there exists a countable set $D \subseteq X$ such that $\overline{D} = X$.

Definition A.15 (Polish Space)**.** A topological space X is called a *Polish space* if it is separable and completely metrizable. That is, there exists a metric d on X which induces the topology of X such that (X, d) is a complete metric space.

A.3 Compactness

Definition A.16 (Closed Set)**.** A subset $F \subseteq X$ is *closed* if its complement $X \backslash F$ is open.

Definition A.17 (Closure)**.** The *closure* of a subset $A \subseteq X$, denoted $\overline{A}$, is the intersection of all closed sets containing A. It is the smallest closed set containing A.

Definition A.18 (Bounded Set)**.** A subset A of a metric space (X, d) is *bounded* if there exists $x \in X$ and $R > 0$ such that $A \subseteq B_R(x)$.

Definition A.19 (Compactness)**.** A subset K of a topological space X is *compact* if every open cover of K has a finite subcover. That is, if $K \subseteq \bigcup_{i \in I} U_i$ where each U_i is open, then there exists a finite subset $J \subseteq I$ such that $K \subseteq \bigcup_{j \in J} U_j$.

Definition A.20 (Sequential Compactness)**.** A subset K of a metric space is *sequentially compact* if every sequence in K has a convergent subsequence whose limit belongs to K.

In metric spaces, compactness and sequential compactness are equivalent.

Definition A.21 (Limit Point)**.** A point $x \in X$ is a *limit point* (or accumulation point) of a set A if every open neighborhood of x contains a point of A distinct from x.

Definition A.22 (Generated Topology)**.** Let X be a set and $\mathcal{S}$ be a collection of subsets of X. The *topology generated by* $\mathcal{S}$ is the smallest topology on X containing $\mathcal{S}$. It consists of all arbitrary unions of finite intersections of elements of $\mathcal{S}$. The elements of $\mathcal{S}$ are called a *subbasis* for the topology.

Example A.23 (Standard Topology on $\mathbb{R}$)**.** Let $X = \mathbb{R}$. The standard topology on $\mathbb{R}$ is the topology generated by the collection of all open intervals (a, b). In fact, this is the same as the topology induced by the Euclidean metric $d(x, y) = |x - y|$. ◊

Example A.24 (Topology of Pointwise Convergence)**.** Let X be the set of all functions $f : [0, 1] \to \mathbb{R}$. The topology of pointwise convergence is the topology generated by sets of the form

$$S_{t,(a,b)} = \{f \in X : a < f(t) < b\}$$

where $t \in [0,1]$ and $a < b$ are real numbers. Convergence in this topology corresponds exactly to pointwise convergence: a sequence $f_n \to f$ if and only if $f_n(t) \to f(t)$ for all $t \in [0,1]$. ◊